6

Flow of viscous fluid

All fluids are viscous. In the case where the viscous effect is minimal, the flow can be treated as an ideal fluid, otherwise the fluid must be treated as a viscous fluid. For example, it is necessary to treat a fluid as a viscous fluid in order to analyse the pressure loss due to a flow, the drag acting on a body in a flow and the phenomenon where flow separates from a body. In this chapter, such fundamental matters are explained to obtain analytically the relation between the velocity, pressure, etc., in the flow of a two-dimensional incompressible viscous fluid.

6.1 Continuity equation

Consider the elementary rectangle of fluid of side dx, side dy and thickness b as shown in Fig. 6.1 (b being measured perpendicularly to the paper). The velocities in the x and y directions are u and v respectively. For the x

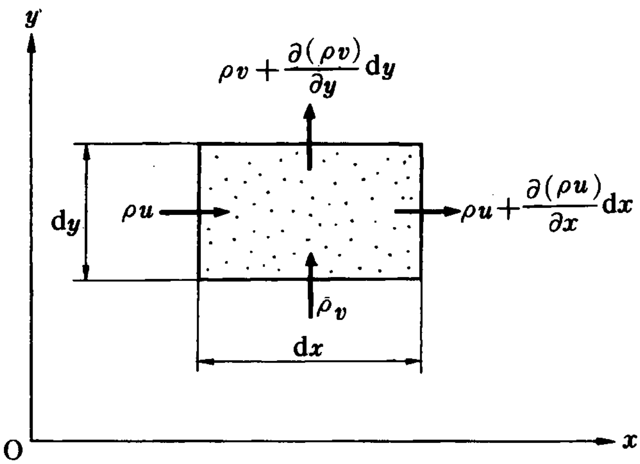


Fig. 6.1 Flow balance in a fluid element

direction, by deducting the outlet mass flow rate from the inlet mass flow rate, the fluid mass stored in the fluid element per unit time can be obtained, i.e.

$$\rho u b\,dy - \left[\rho u + \frac{\partial(\rho u)}{\partial x}dx\right] b\,dy = -\frac{\partial(\rho u)}{\partial x} b\,dx\,dy$$

Similarly, the fluid mass stored in it per unit time in the y direction is

$$-\frac{\partial(\rho v)}{\partial y} b\,dx\,dy$$

The mass of fluid element ($\rho b\,dx\,dy$) ought to increase by $\partial(\rho b\,dx\,dy/\partial t)$ in unit time by virtue of this stored fluid. Therefore, the following equation is obtained:

$$-\frac{\partial(\rho u)}{\partial x} b\,dx\,dy - \frac{\partial(\rho v)}{\partial y} b\,dx\,dy = \frac{\partial(\rho b\,dx\,dy)}{\partial t}$$

or

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0 \tag{6.1}$$ [1]

Equation (6.1) is called the continuity equation. This equation is applicable to the unsteady flow of a compressible fluid. In the case of steady flow, the first term becomes zero.

For an incompressible fluid, ρ is constant, so the following equation is obtained:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{6.2}$$ [1]

This equation is applicable to both steady and unsteady flows.

In the case of an axially symmetric flow as shown in Fig. 6.2, eqn (6.2) becomes, using cylindrical coordinates,

$$\frac{\partial u}{\partial x} + \frac{1}{r}\frac{\partial(rv)}{\partial r} = 0 \tag{6.3}$$

As the continuity equation is independent of whether the fluid is viscous or not, the same equation is applicable also to an ideal fluid.

6.2 Navier–Stokes equation

Consider an elementary rectangle of fluid of side dx, side dy and thickness b as shown in Fig. 6.3, and apply Newton's second law of motion. Where the

[1] $\partial u/\partial x + \partial v/\partial y + \partial w/\partial z$ is generally called the divergence of vector V (whose components x, y, z are u, v, w) and is expressed as div V or ∇V. If we use this, eqns (6.1) and (6.2) (two-dimensional flow, so $w = 0$) are expressed respectively as the following equations:

$$\frac{\partial \rho}{\partial t} + \mathrm{div}(\rho V) = 0 \quad \text{or} \quad \frac{\partial \rho}{\partial t} + \nabla(\rho V) = 0 \tag{6.1'}$$

$$\mathrm{div}(\rho V) = 0 \quad \text{or} \quad \nabla(\rho V) = 0 \tag{6.2'}$$

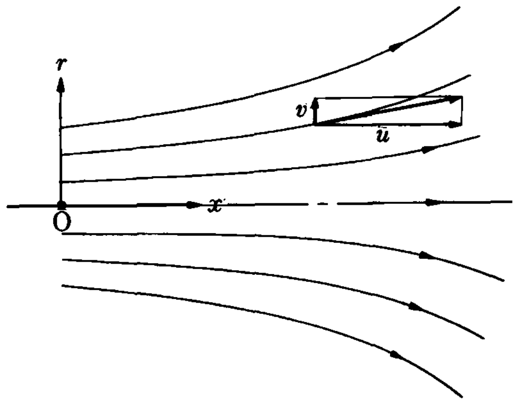


Fig. 6.2 Axially symmetric flow

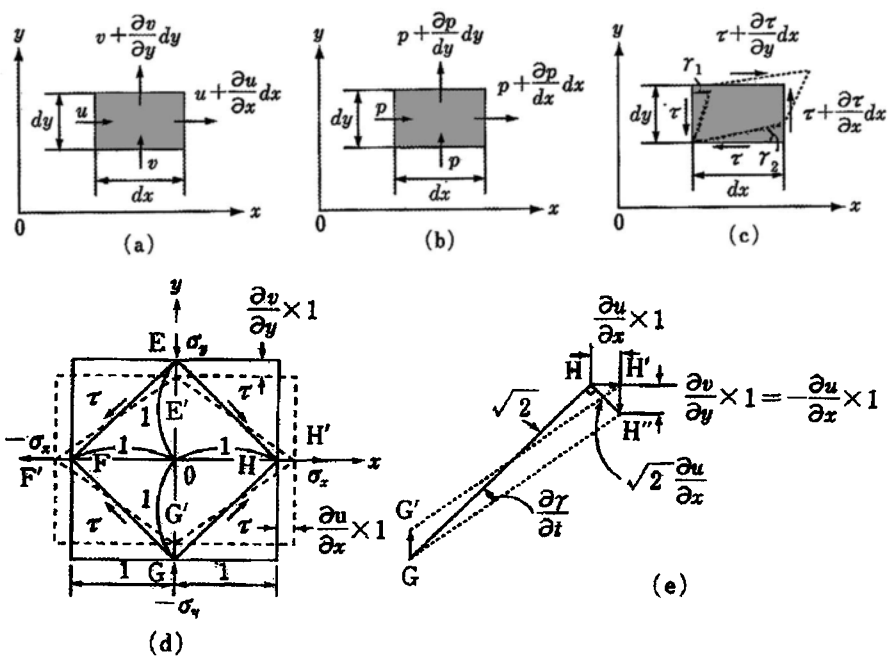


Fig. 6.3 Balance of forces on a fluid element: (a) velocity; (b) pressure; (c) angular deformation; (d) relation between tensile stress and shearing stress by elongation transformation of x direction; (e) velocity of angular deformation by elongation and contraction

forces acting on this element are $F(F_x, F_y)$, the following equations are obtained for the x and y axes respectively:

$$\left.\begin{aligned} \rho b\,\mathrm{d}x\,\mathrm{d}y\frac{\mathrm{d}u}{\mathrm{d}t} &= F_x \\ \rho b\,\mathrm{d}x\,\mathrm{d}y\frac{\mathrm{d}v}{\mathrm{d}t} &= F_y \end{aligned}\right\} \tag{6.4}$$

The left-hand side of eqn (6.4) expresses the inertial force which is the product of the mass and acceleration of the fluid element. The change in velocity of this element is brought about both by the movement of position and by the progress of time. So the velocity change du at time dt is expressed by the following equation:

$$\mathrm{d}u = \frac{\partial u}{\partial t}\mathrm{d}t + \frac{\partial u}{\partial x}\mathrm{d}x + \frac{\partial u}{\partial y}\mathrm{d}y$$

Therefore,

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial u}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}$$

Substituting this into eqn (6.4),

$$\left.\begin{aligned} \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) b\,\mathrm{d}x\,\mathrm{d}y = F_x \\ \rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) b\,\mathrm{d}x\,\mathrm{d}y = F_y \end{aligned}\right\} \tag{6.5}$$

Next, the force F acting on the elements comprises the body force $F_B(B_x, B_y)$, pressure force $F_p(P_x, P_y)$ and viscous force $F_s(S_x, S_y)$. In other words, F_x and F_y are expressed by the following equation:

$$\begin{aligned} F_x = B_x + P_x + S_x \\ F_y = B_y + P_y + S_y \end{aligned} \tag{6.6}$$

Body force $F_b(B_x, B_y)$

(These forces act directly throughout the mass, such as the gravitational force, the centrifugal force, the electromagnetic force, etc.) Putting X and Y as the x and y axis components of such body forces acting on the mass of fluid, then

$$\left.\begin{aligned} B_x = X\rho b\,\mathrm{d}x\,\mathrm{d}y \\ B_y = Y\rho b\,\mathrm{d}x\,\mathrm{d}y \end{aligned}\right\} \tag{6.7}$$

For the gravitational force, $X = 0$, $Y = -g$.

Pressure force $F_p(P_x, P_y)$

Here,

$$\left.\begin{aligned} P_x &= pb\,\mathrm{d}y - \left(p + \frac{\partial p}{\partial x}\mathrm{d}x\right) b\,\mathrm{d}y = -\frac{\partial p}{\partial x} b\,\mathrm{d}x\,\mathrm{d}y \\ P_y &= -\frac{\partial p}{\partial y} b\,\mathrm{d}x\,\mathrm{d}y \end{aligned}\right\} \tag{6.8}$$

Viscous force $F_s(S_x, S_y)$

Force in the x direction due to angular deformation, S_{x1} Putting the strain of the small element of fluid $\gamma = \gamma_1 + \gamma_2$, the corresponding stress is expressed as $\tau = \mu\, \partial\gamma/\partial t$:

$$\tau = \mu\frac{\partial\gamma}{\partial t} = \mu\left(\frac{\partial\gamma_1}{\partial t} + \frac{\partial\gamma_2}{\partial t}\right) = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

So,

$$S_{x1} = \frac{\partial\tau}{\partial y} b\,\mathrm{d}x\,\mathrm{d}y = \mu\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial x\,\partial y}\right) b\,\mathrm{d}x\,\mathrm{d}y = \mu\left(\frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial x^2}\right) b\,\mathrm{d}x\,\mathrm{d}y \quad (6.9)$$

Force in the x direction due to elongation transformation, S_{x2} Consider the rhombus EFGH inscribed in a cubic fluid element ABCD of unit thickness as shown in Fig. 6.3(d), which shows that an elongated flow to x direction is a contracted flow to y direction. This deformation in the x and y directions produces a simple angular deformation seen in the rotation of the faces of the rhombus.

Now, calculating the deformation per unit time, the velocity of angular deformation $\partial\gamma/\partial t$ becomes as seen from Fig. 6.3(e).

$$\frac{\partial\gamma}{\partial t} = \frac{\sqrt{2}\dfrac{\partial u}{\partial x}}{\sqrt{2}} = \frac{\partial u}{\partial x}$$

Therefore, a shearing stress τ acts on the four faces of the rhombus EFGH.

$$\tau = \mu\frac{\partial\gamma}{\partial t} = \mu\frac{\partial u}{\partial x}$$

For equilibrium of the force on face EG due to the tensile stress σ_x and the shear forces on EH and HG due to τ

$$\sigma_x = 2 \times \sqrt{2}\tau\cos 45^\circ = 2\tau$$

$$\sigma_x = 2\mu\frac{\partial u}{\partial x}$$

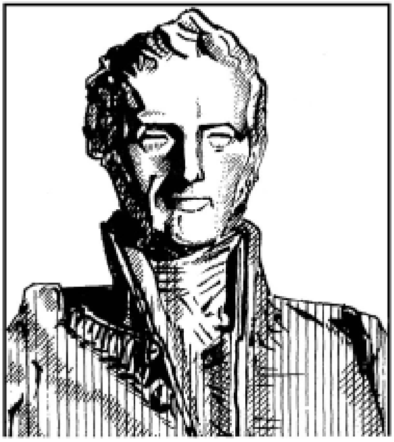
Considering the fluid element having sides $\mathrm{d}x, \mathrm{d}y$ and thickness b, the tensile stress in the x direction on the face at distance $\mathrm{d}x$ becomes $\sigma_x + \dfrac{\partial\sigma_x}{\partial x}\,\mathrm{d}x$. This stress acts on the face of area $b\,\mathrm{d}y$, so the force σ_{x2} in the x direction is

$$S_{x2} = -(\sigma_x)_x b\,\mathrm{d}y + (\sigma_x)_{x+\mathrm{d}x} b\,\mathrm{d}y = \left\{-\sigma_x + \left(\sigma_x + \frac{\partial\sigma_x}{\partial x}\,\mathrm{d}x\right)\right\} b\,\mathrm{d}y$$

$$= \frac{\partial\sigma_x}{\partial x} b\,\mathrm{d}x\,\mathrm{d}y = 2\mu\frac{\partial^2 u}{\partial x^2} b\,\mathrm{d}x\,\mathrm{d}y \quad (6.10)$$

Therefore,

$$\left.\begin{aligned} S_x &= S_{x1} + S_{x2} = \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) b\,\mathrm{d}x\,\mathrm{d}y \\ S_y &= \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) b\,\mathrm{d}x\,\mathrm{d}y \end{aligned}\right\} \quad (6.11)$$



Louis Marie Henri Navier (1785–1836)
Born in Dijon, France. Actively worked in the educational and bridge engineering fields. His design of a suspension bridge over the River Seine in Paris attracted public attention. In analysing fluid movement, thought of an assumed force by repulsion and absorption between neighbouring molecules in addition to the force studied by Euler to find the equation of motion of fluid. Thereafter, through research by Cauchy, Poisson and Saint-Venant, Stokes derived the present equations, including viscosity.

Substituting eqns (6.7), (6.8) and (6.10) into eqn (6.5), the following equation is obtained:

$$\left.\begin{aligned}\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)&=\rho X-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)\\ \underbrace{\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)}_{\text{Inertia term}}&=\underbrace{\rho Y}_{\text{Body force term}}-\underbrace{\frac{\partial p}{\partial y}}_{\text{Pressure term}}+\mu\underbrace{\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)}_{\text{Viscous term}}\end{aligned}\right\} \tag{6.12}$$

These equations are called the Navier–Stokes equations. In the inertia term, the rates of velocity change with position and

$$\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)\quad\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)$$

and so are called the convective accelerations.

In the case of axial symmetry, when cylindrical coordinates are used, eqns (6.12) become the following equations:

$$\left.\begin{aligned}\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial r}\right)&=\rho X-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{\partial^2 u}{\partial r^2}\right)\\ \rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial r}\right)&=\rho R-\frac{\partial p}{\partial r}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{1}{r}\frac{\partial v}{\partial r}-\frac{v}{r^2}+\frac{\partial^2 v}{\partial r^2}\right)\end{aligned}\right\} \tag{6.13}$$

where R is the r direction component of external force acting on the fluid of unit mass.

The vorticity ζ is

$$\zeta=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial r} \tag{6.14}$$

and the shearing stress is

$$\tau=-\mu\left(\frac{\partial u}{\partial r}+\frac{\partial v}{\partial x}\right) \tag{6.15}$$

The continuity equation (6.3), along with equation (6.15), are convenient for analysing axisymmetric flow in pipes.

Now, omitting the body force terms, eliminating the pressure terms by partial differentiation of the upper equation of eqn (6.12) by y and the lower equation by x, and then rewriting these equations using the equation of vorticity (4.7), the following equation is obtained:

$$\rho\left(\frac{\partial \zeta}{\partial t}+u\frac{\partial \zeta}{\partial x}+v\frac{\partial \zeta}{\partial y}\right)=\mu\left(\frac{\partial^2 \zeta}{\partial x^2}+\frac{\partial^2 \zeta}{\partial y^2}\right) \tag{6.16}$$

For ideal flow, $\mu = 0$, so the right-hand side of eqn (6.11) becomes zero. Then it is clear that the vorticity does not change in the ideal flow process. This is called the vortex theory of Helmholz.

Now, non-dimensionalise the above using the representative size l and the representative velocity U:

$$\begin{aligned} x^* &= x/l \quad y^* = y/l \\ u^* &= u/U \quad v^* = v/U \\ t^* &= tU/l \\ \zeta^* &= \partial v^*/\partial x^* - \partial u^*/\partial y^* \\ Re &= \rho U l/\mu \end{aligned} \tag{6.17}$$

Using these equations rewrite eqn (6.16) to obtain the following equation:

$$\frac{\partial \zeta^*}{\partial t^*}+u^*\frac{\partial \zeta^*}{\partial x^*}+v^*\frac{\partial \zeta^*}{\partial y^*}=\frac{1}{Re}\left(\frac{\partial^2 \zeta^*}{\partial x^{*2}}+\frac{\partial^2 \zeta^*}{\partial y^{*2}}\right) \tag{6.18}$$

Equation (6.18) is called the vorticity transport equation. This equation shows that the change in vorticity due to fluid motion equals the diffusion of vorticity by viscosity. The term $1/Re$ corresponds to the coefficient of diffusion. Since a smaller Re means a larger coefficient of diffusion, the diffusion of vorticity becomes larger, too.

6.3 Velocity distribution of laminar flow

In the Navier–Stokes equations, the convective acceleration in the inertial term is non-linear[2]. Hence it is difficult to obtain an analytical solution for general flow. The strict solutions obtained to date are only for some special flows. Two such examples are shown below.

6.3.1 Flow between parallel plates

Let us study the flow of a viscous fluid between two parallel plates as shown in Fig. 6.4, where the flow has just passed the inlet length (see Section 7.1)

[2] The case where an equation is not a simple equation for the unknown function and its partial differential function is called non-linear.

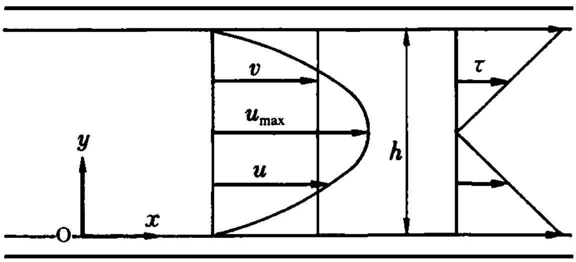


Fig. 6.4 Laminar flow between parallel plates

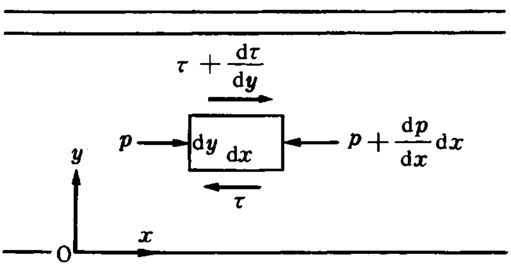
where it had flowed in the laminar state. For the case of a parallel flow like this, the Navier–Stokes equation (6.12) is extremely simple as follows:

1. As the velocity is only u since $v = 0$, it is sufficient to use only the upper equation.
2. As this flow is steady, u does not change with time, so $\partial u/\partial t = 0$.
3. As there is no body force, $\rho X = 0$.
4. As this flow is uniform, u does not change with position, so $\partial u/\partial x = 0$ and $\partial^2 u/\partial x^2 = 0$.
5. Since $v = 0$, the lower equation of (6.12) simply expresses the hydrostatic pressure variation and has no influence in the x direction.

So, the upper equation of eqn (6.12) becomes

$$\mu \frac{d^2 u}{dy^2} = \frac{dp}{dx} \tag{6.19}$$ [3]

[3] Consider the balance of forces acting on the respective faces of an assumed small volume $dx\,dy$ (of unit width) in a fluid.



Forces acting on a small volume between parallel plates

Since there is no change of momentum between the two faces, the following equation is obtained:

$$p\,dy - \left(p + \frac{dp}{dx}dx\right)dy - \tau\,dx + \left(\tau + \frac{d\tau}{dy}dy\right)dx = 0$$

Therefore

$$\frac{d\tau}{dy} = \frac{dp}{dx}$$

and

$$\tau = \mu\frac{du}{dy} \quad \text{since} \quad \mu\frac{d^2u}{dy^2} = \frac{dp}{dx} \tag{6.19'}$$

By integrating the above equation twice about y, the following equation is obtained:

$$u = \frac{1}{2\mu}\frac{\mathrm{d}p}{\mathrm{d}x}y^2 + c_1 y + c_2 \tag{6.20}$$

Using $u = 0$ as the boundary condition at $y = 0$ and h, c_1 and c_2 are found as follows:

$$u = -\frac{1}{2\mu}\frac{\mathrm{d}p}{\mathrm{d}x}(h - y)y \tag{6.21}$$

It is clear that the velocity distribution now forms a parabola.

At $y = h/2$, $\mathrm{d}u/\mathrm{d}y = 0$, so u becomes u_{max}:

$$u_{max} = -\frac{1}{8\mu}\frac{\mathrm{d}p}{\mathrm{d}x}h^2 \tag{6.22}$$

The volumetric flow rate Q becomes

$$Q = \int_0^h u\,\mathrm{d}y = -\frac{1}{12\mu}\frac{\mathrm{d}p}{\mathrm{d}x}h^3 \tag{6.23}$$

From this equation, the mean velocity v is

$$v = \frac{Q}{h} = -\frac{1}{12\mu}\frac{\mathrm{d}p}{\mathrm{d}x}h^2 = \frac{1}{1.5}u_{max} \tag{6.24}$$

The shearing stress τ due to viscosity becomes

$$\tau = \mu\frac{\mathrm{d}u}{\mathrm{d}y} = -\frac{1}{2}\frac{\mathrm{d}p}{\mathrm{d}x}(h - 2y) \tag{6.25}$$

The velocity and shearing stress distribution are shown in Fig. 6.4.

Figure 6.5 is a visualised result using the hydrogen bubble method. It is clear that the experimental result coincides with the theoretical result.

Putting l as the length of plate in the flow direction and Δp as the pressure difference, and integrating in the x direction, the following relation is obtained:

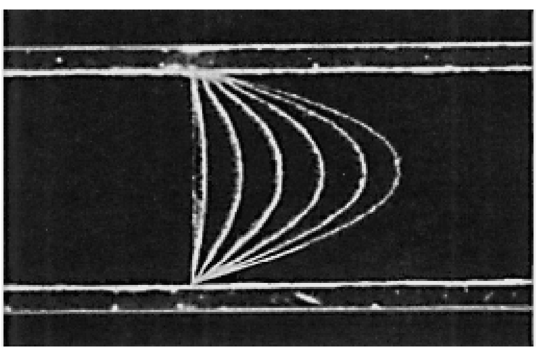


Fig. 6.5 Flow, between parallel plates (hydrogen bubble method), of water, velocity 0.5 m/s, $Re = 140$

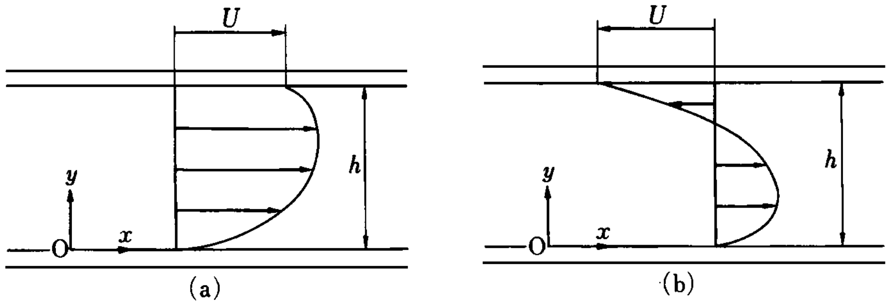


Fig. 6.6 Couette–Poiseuille flow[4]

$$-\frac{dp}{dx}=\frac{\Delta p}{l} \tag{6.26}$$

Substituting this equation into eqn (6.23) gives

$$Q=\frac{\Delta p h^3}{12\mu l} \tag{6.27}$$

As shown in Fig. 6.6, in the case where the upper plate moves in the x direction at constant speed U or $-U$, from the boundary conditions of $u=0$ at $y=0$ and $u=U$ at $y=h$, c_1 and c_2 in eqn (6.20) can be determined. Thus

$$u=\frac{\Delta p}{2\mu l}(h-y)y\pm\frac{Uy}{h} \tag{6.28}$$

Then, the volumetric flow rate Q is as follows:

$$Q=\int_0^h u\,dy=\frac{\Delta p h^3}{12\mu l}\pm\frac{Uh}{2} \tag{6.29}$$

6.3.2 Flow in circular pipes

A flow in a long circular pipe is a parallel flow of axial symmetry (Fig. 6.7). In this case, it is convenient to use the Navier–Stokes equation (6.13) using cylindrical coordinates. Under the same conditions as in the previous section (6.3.1), simplify the upper equation in equation (6.13) to give

$$\frac{dp}{dx}=\mu\left(\frac{d^2u}{dr^2}+\frac{1}{r}\frac{du}{dr}\right) \tag{6.30}$$

Integrating,

$$u=\frac{1}{4\mu}\frac{dp}{dx}r^2+c_1\log r+c_2 \tag{6.31}$$

According to the boundary conditions, since the velocity at $r=0$ must be finite $c_1=0$ and c_2 is determined when $u=0$ at $r=r_0$:

[4] Assume a viscous fluid flowing between two parallel plates; fix one of the plates and move the other plate at velocity U. The flow in this case is called Couette flow. Then, fix both plates, and have the fluid flow by the differential pressure. The flow in this case is called two-dimensional Poiseuille flow. The combination of these two flows as shown in Fig. 6.6 is called Couette–Poiseuille flow.

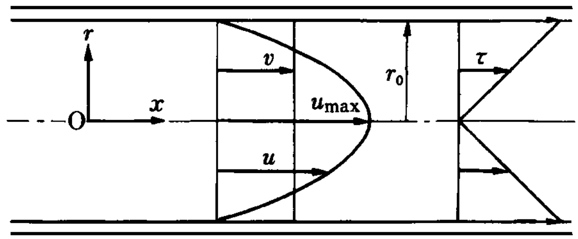


Fig. 6.7 Laminar flow in a circular pipe

$$u = -\frac{1}{4\mu}\frac{dp}{dx}(r_0^2 - r^2) \tag{6.32}$$

From this equation, it is clear that the velocity distribution forms a paraboloid of revolution with u_{max} at $r = 0$:

$$u_{max} = -\frac{1}{4\mu}\frac{dp}{dx}r_0^2 \tag{6.33}$$

The volumetric flow rate passing pipe Q becomes

$$Q = \int_0^{r_0} 2\pi r u \, dr = -\frac{\pi r_0^4}{8\mu}\frac{dp}{dx} \tag{6.34}$$

From this equation, the mean velocity v is

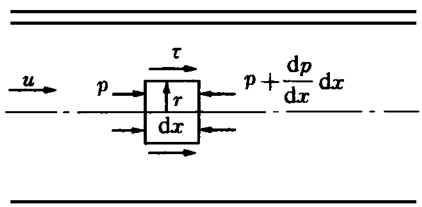
$$v = \frac{Q}{\pi r_0^2} = -\frac{r_0^2}{8\mu}\frac{dp}{dx} = \frac{1}{2}u_{max} \tag{6.35}$$

The shear stress due to the viscosity is,

$$\tau = -\mu\frac{du}{dr} = -\frac{1}{2}\frac{dp}{dx}r \tag{6.36}$$[5]

The velocity distribution and the shear distribution are shown in Fig. 6.7.

[5] Equation (6.36) can be deduced by the balance of forces. From the diagram



Force acting on a cylindrical element in a round pipe

$$-\pi r^2\frac{dp}{dx} + 2\pi r\tau \, dx = 0$$
$$\tau = \mu\frac{du}{dr}$$

(Since $du/dr < 0$, τ is negative, i.e. leftward.)

Thus

$$\frac{du}{dr} = \frac{1}{2\mu}\frac{dp}{dx}r \tag{6.36$'$}$$

is obtained.

Gotthilf Heinrich Ludwig Hagen (1797–1884)
German hydraulic engineer. Conducted experiments on the relation between head difference and flow rate. Had water mixed with sawdust flow in a brass pipe to observe its flowing state at the outlet. Was yet to discover the general similarity parameter including the viscosity, but reported that the transition from laminar to turbulent flow is connected with tube diameter, flow velocity and water temperature.

A visualisation result using the hydrogen bubble method is shown in Fig. 6.8.

Putting the pressure drop in length l as Δp, the following equation is obtained from eqn (6.33):

$$\Delta p = \frac{128\mu l Q}{\pi d^4} = \frac{32\mu l v}{d^2} \tag{6.37}$$

This relation was discovered independently by Hagen (1839) and Poiseuille (1841), and is called the Hagen–Poiseuille formula. Using this equation, the viscosity of liquid can be obtained by measuring the pressure drop Δp.

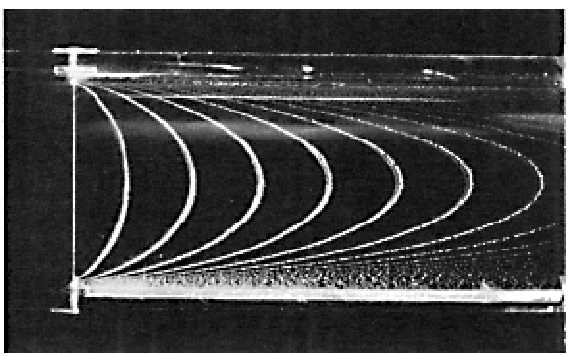
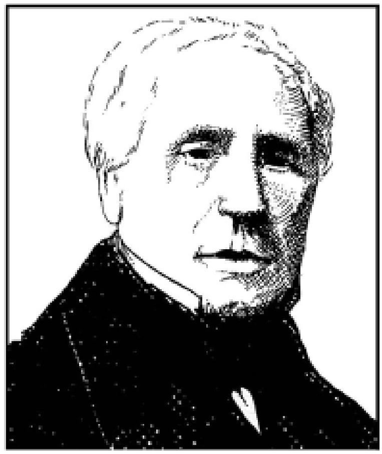


Fig. 6.8 Velocity distribution, in a circular pipe (hydrogen bubble method), of water, velocity 2.4 m/s, $Re = 195$



Jean Louis Poiseuille (1799–1869)
French physician and physicist. Studied the pumping power of the heart, the movement of blood in vessels and capillaries, and the resistance to flow in a capillary. In his experiment on a glass capillary (diameter 0.029–0.142 mm) he obtained the experimental equation that the flow rate is proportional to the product of the difference in pressure by a power of 4 of the pipe inner diameter, and in inverse proportion to the tube length.

6.4 Velocity distribution of turbulent flow

As stated in Section 4.4, flow in a round pipe is stabilised as laminar flow whenever the Reynolds number *Re* is less than 2320 or so, but the flow becomes turbulent through the transition region as *Re* increases. In turbulent flow, as observed in the experiment where Reynolds let coloured liquid flow, the fluid particles have a velocity minutely fluctuating in an irregular short cycle in addition to the timewise mean velocity. By measuring the flow with a hot-wire anemometer, the fluctuating velocity as shown in Fig. 6.9 can be recorded.

For two-dimensional flow, the velocity is expressed as follows:

$$u = \overline{u} + u' \qquad v = \overline{v} + v'$$

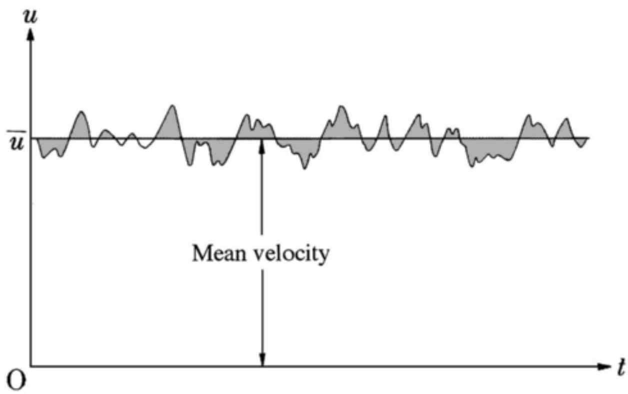


Fig. 6.9 Turbulence

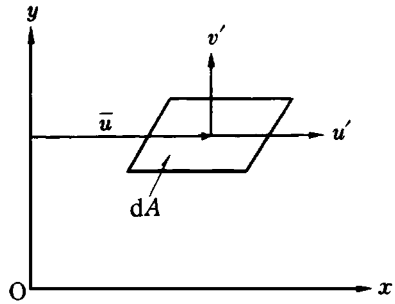


Fig. 6.10 Momentum transport by turbulence

where $\bar{u}$ and $\bar{v}$ are the timewise mean velocities and u' and v' are the fluctuating velocities.

Now, consider the flow at velocity u in the x direction as the flow between two flat plates (Fig. 6.10), so $u = \bar{u} + u'$ but $v = v'$.

The shearing stress τ of a turbulent flow is now the sum of laminar flow shearing stress (viscous friction stress) τ_l, which is the frictional force acting between the two layers at different velocities, and so-called turbulent shearing stress τ_t, where numerous rotating molecular groups (eddies) mix with each other. Thus

$$\tau = \tau_l + \tau_t \tag{6.38}$$

Now, let us examine the turbulent shearing stress only. As shown in Fig. 6.10, the fluid which passes in unit time in the y direction through minute area $\mathrm{d}A$ parallel to the x axis is $\rho v' \mathrm{d}A$. Since this fluid is at relative velocity u', the momentum is $\rho v' \mathrm{d}A u'$. By the movement of this fluid, the upper fluid increases its momentum per unit area by $\rho u'v'$ in the positive direction of x per unit time. Therefore, a shearing stress develops on face $\mathrm{d}A$. In other words, it is found that the shearing stress due to the turbulent flow is proportional to $\rho u'v'$. Reynolds, by substituting $u = \bar{u} + u'$, $v = \bar{v} + v'$ into the Navier–Stokes equation, performed an averaging operation over time and derived $-\rho\overline{u'v'}$ as a shearing stress in addition to that due to the viscosity.

Thus

$$\tau_t = -\rho\overline{u'v'} \tag{6.39}$$

where τ_t is the stress developed by the turbulent flow, which is called the Reynolds stress. As can be seen from this equation, the correlation[6] $\overline{u'v'}$ of

[6] In general, the mean of the product of a large enough number of two kinds of quantities is called the correlation. Whenever this value is large, the correlation is said to be strong. In studying turbulent flow, one such correlation is the timewise mean of the products of fluctuating velocities in two directions. Whenever this value is large, it indicates that the velocity fluctuations in two directions fluctuate similarly timewise. Whenever this value is near zero, it indicates that the correlation is small between the fluctuating velocities in two directions. And whenever this value is negative, it indicates that the fluctuating velocities fluctuate in reverse directions to each other.

Ludwig Prandtl (1875–1953)
Born in Germany, Prandtl taught at Hanover Engineering College and then Göttingen University. He successfully observed, by using the floating tracer method, that the surface of bodies is covered with a thin layer having a large velocity gradient, and so advocated the theory of the boundary layer. He is called the creator of modern fluid dynamics. Furthermore, he taught such famous scholars as Blasius and Kármán. Wrote *The Hydrology.*

---

the fluctuating velocity is necessary for computing the Reynolds stress. Figure 6.11 shows the shearing stress in turbulent flow between parallel flat plates.

Expressing the Reynolds stress as follows as in the case of laminar flow

$$\tau_t = \rho\nu\frac{d\bar{u}}{dy} \tag{6.40}$$

produces the following as the shearing stress in turbulent flow:

$$\tau = \tau_l + \tau_t = \rho(\nu + \nu_t)\frac{d\bar{u}}{dy} \tag{6.41}$$

This ν_t is called the turbulent kinematic viscosity. ν_t is not the value of a physical property dependent on the temperature or such, but a quantity fluctuating according to the flow condition.

Prandtl assumed the following equation in which, for rotating small parcels

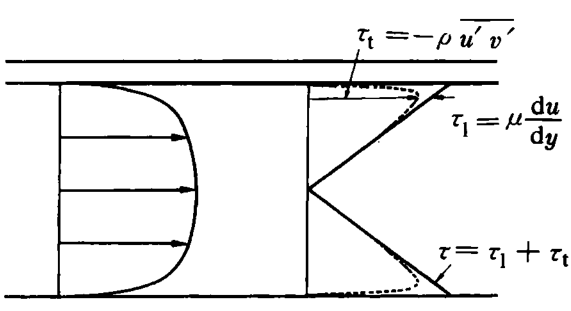


Fig. 6.11 Distribution of shearing stresses of flow between parallel flat plates (enlarged near the wall)

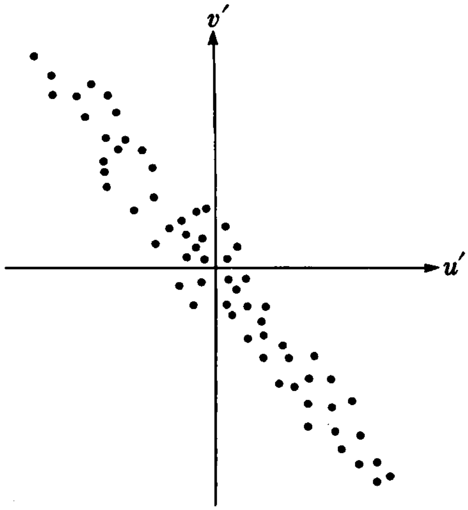


Fig. 6.12 Correlation of u' and v'

of fluid of turbulent flow (eddies) travelling average length, the eddies assimilate the character of other eddies by collisions with them:

$$|u'| \simeq |v'| = l\left|\frac{d\bar{u}}{dy}\right| \tag{6.42}$$

Prandtl called this l the mixing length.

According to the results of turbulence measurements for shearing flow, the distributions of u' and v' are as shown in Fig. 6.12, where $u'v'$ has a large probability of being negative. Furthermore, the mixing length is redefined as follows, including the constant of proportionality:

$$-\overline{u'v'} = l^2\left(\frac{d\bar{u}}{dy}\right)^2$$

so that

$$\tau_t = -\rho\overline{u'v'} = \rho l^2\left(\frac{d\bar{u}}{dy}\right)^2 \tag{6.43}$$ [7]

The relation in eqn (6.43) is called Prandtl's hypothesis on mixing length, which is widely used for computing the turbulence shearing stress. Mixing length l is not the value of a physical property but a fluctuating quantity depending on the velocity gradient and the distance from the wall. This

[7] According to the convention that the symbol for shearing stress is related to that of velocity gradient, it is described as follows:

$$\tau_t = \rho l^2\left|\frac{d\bar{u}}{d\bar{y}}\right|\frac{d\bar{u}}{dy}$$

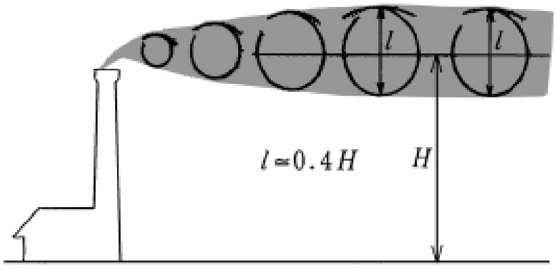


Fig. 6.13 Smoke vortices from a chimney

introduction of l is replaced in eqn (6.40) to produce a computable fluctuating quantity.

At this stage, however, Prandtl came to a standstill. That is, unless some concreteness was given to l, no further development could be undertaken. At a loss, Prandtl went outdoors to refresh himself. In the distance there stood some chimneys, the smoke from which was blown by a breeze as shown in Fig. 6.13. He noticed that the vortices of smoke near the ground were not so large as those far from the ground. Subsequently, he found that the size of the vortex was approximately 0.4 times the distance between the ground and the centre of the vortex. On applying this finding to a turbulent flow, he derived the relation $l = 0.4y$. By substituting this relation into eqn (6.43), the following equation was obtained:

$$\frac{d\bar{u}}{dy} = \frac{1}{0.4y}\sqrt{\frac{\tau_t}{\rho}} \tag{6.44}$$

Next, in an attempt to establish τ_t, he focused his attention on the flow near the wall. There, owing to the presence of wall, a thin layer δ_0 developed where turbulent mixing is suppressed and the effect of viscosity dominates as shown in Fig. 6.14. This extremely thin layer is called the viscous sublayer.[8] Here, the velocity distribution can be regarded as the same as in laminar flow, and ν_t in eqn (6.41) becomes almost zero. Assuming τ_0 to be the shearing stress acting on the wall, then so far as this section is concerned:

$$\tau_0 = \mu\frac{du}{dy} = \mu\frac{u}{y} \quad (y \le \delta_0)$$

or

$$\frac{\tau_0}{\rho} = \nu\frac{u}{y} \tag{6.45}$$

$\sqrt{\tau_0/\rho}$ has the dimension of velocity, and is called the friction velocity,

[8] Until some time ago, this layer had been conceived as a laminar flow and called the laminar sublayer, but recently research on visualisation by Kline at Stanford University and others found that the turbulent fluctuation parallel to the wall (bursting process) occurred here, too. Consequently, it is now called the viscous sublayer.

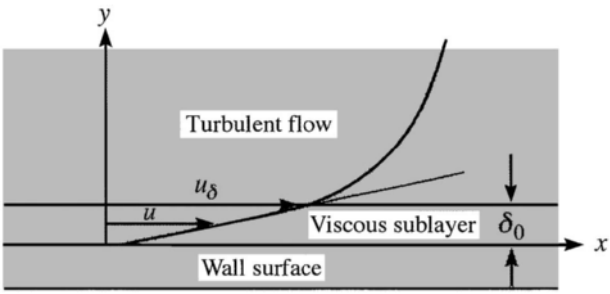


Fig. 6.14 Viscous sublayer

symbol v_* (v star). Substituting, eqn (6.45) becomes:

$$\frac{u}{v_*} = \frac{v_* y}{\nu} \tag{6.46}$$

Putting $u = u_\delta$ whenever $y = \delta_0$ gives

$$\frac{u_\delta}{v_*} = \frac{v_* \delta_0}{\nu} = R_\delta \tag{6.47}$$

where R_δ is a Reynolds number.

Next, since turbulent flow dominates in the neighbourhood of the wall beyond the viscous sublayer, assume $\tau_t = \tau_0$,[9] and integrate eqn (6.44):

$$\frac{\bar{u}}{v_*} = 2.5 \ln y + c \tag{6.48}$$

Using the relation $\bar{u} = u_\delta$ when $y = \delta_0$,

$$c = \frac{u_\delta}{v_*} - 2.5 \ln \delta_0 = R_\delta - 2.5 \ln \delta_0 \tag{6.49}$$

Substituting the above into eqn (6.48) gives

$$\frac{\bar{u}}{v_*} = 2.5 \ln\left(\frac{y}{\delta_0}\right) + R_\delta$$

Using the relation in eqn (6.47),

$$\frac{\bar{u}}{v_*} = 2.5 \ln\left(\frac{v_* y}{\nu}\right) + A \tag{6.50}$$

If $\bar{u}/v_*$, is plotted against $\log_{10}(v_* y/\nu)$, it turns out as shown in Fig. 6.15 giving $A = 5.5$.[10]

[9] $\tau_t = \tau_0$ was the assumption for the case in the neighbourhood of the wall, and this equation is reasonably applicable when tested off the wall in the direction towards the centre. (Goldstein, S., *Modern Developments in Fluid Dynamics*, (1965), 336, Dover, New York).

[10] It may also be expressed as $\bar{u}/v_* = u^+$, $v_* y/\nu = y^+$.

Theodor von Kármán (1881–1963)
Studied at the Royal Polytechnic Institute of Budapest, and took up teaching positions at Göttingen University, the Polytechnic Institute of Aachen and California Institute of Technology. Beginning with the study of vortices in the flow behind a cylinder, known as the Kármán vortex street, he left many achievements in fluid dynamics including drag on a body and turbulent flow. Wrote *Aerodynamics: Selected Topics in the Light of Their Historical Development.*

$$\frac{\bar{u}}{v_*} = 5.75 \log\left(\frac{v_* y}{\nu}\right) + 5.5 \tag{6.51}$$

This equation is considered applicable only in the neighbourhood of the wall from the viewpoint of its derivation. As seen from Fig. 6.15, however, it was found to be applicable up to the pipe centre from the comparison with the experimental results. This is called the logarithmic velocity distribution, and it is applicable to any value of *Re*.

In addition, Prandtl separately derived through experiment the following

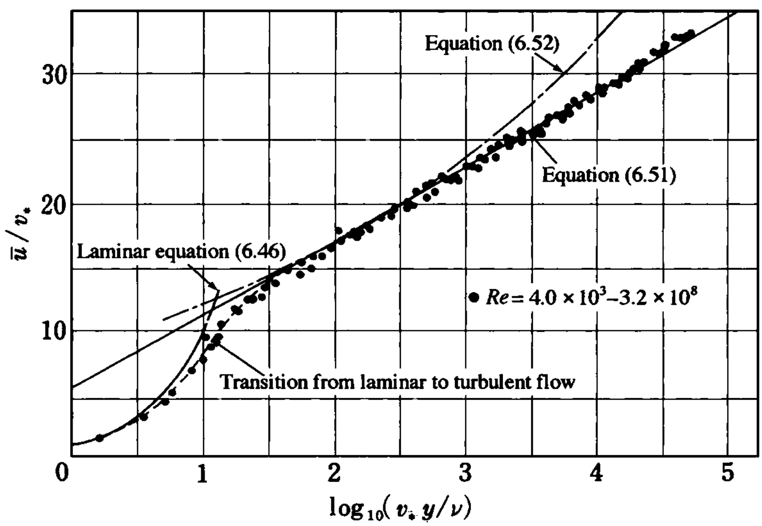


Fig. 6.15 Velocity distribution in a circular pipe (experimental values by Reichardt)

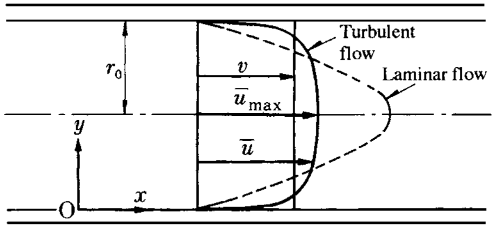


Fig. 6.16 Velocity distribution of turbulent flow

equation of an exponential function as the velocity distribution of a turbulent flow in a circular pipe as shown in Fig. 6.16:

$$\frac{\overline{u}}{\overline{u}_{max}} = \left(\frac{y}{r_0}\right)^{1/n} \quad (0 \leq y \leq r_0) \tag{6.52}$$

n changes according to Re, and is 7 when $Re = 1 \times 10^5$. Since many cases are generally for flows in this neighbourhood, the equation where $n = 7$ is frequently used. This equation is called the Kármán–Prandtl 1/7 power law.[11] Furthermore, there is an experimental equation[12] of $n = 3.45Re^{0.07}$. v/u_{max} is 0.8–0.88

Figure 6.16 also shows the overlaid velocity distributions of laminar and turbulent flows whose average velocities are equal.

Most flows we see daily are turbulent flows, which are important in such applications as heat transfer and mixing. Alongside progress in measuring technology, including visualisation techniques, hot-wire anemometry and laser Doppler velocimetry, and computerised numerical computation, much research is being conducted to clarify the structure of turbulent flow.

6.5 Boundary layer

If the movement of fluid is not affected by its viscosity, it could be treated as the flow of ideal fluid and the viscosity term of eqn (6.11) could be omitted. Therefore, its analysis would be easier. The flow around a solid, however, cannot be treated in such a manner because of viscous friction. Nevertheless, only the very thin region near the wall is affected by this friction. Prandtl identified this phenomenon and had the idea to divide the flow into two regions. They are:

1. the region near the wall where the movement of flow is controlled by the frictional resistance; and
2. the other region outside the above not affected by the friction and, therefore, assumed to be ideal fluid flow.

The former is called the boundary layer and the latter the main flow.

[11] Schlichting, H., *Boundary Layer Theory*, (1968), 563, McGraw-Hill, New York.
[12] Itaya, M, *Bulletin of JSME*, 7-26, (1941–2), III-25.

This idea made the computation of frictional drag etc. acting on a body or a channel relatively easy, and thus enormously contributed to the progress of fluid mechanics.

6.5.1 Development of boundary layer

As shown in Fig. 6.17, at a location far from a body placed in a flow, the flow has uniform velocity U without a velocity gradient. On the face of the body the flow velocity is zero with absolutely no slip. For this reason, owing to the effect of friction the flow velocity near the wall varies continuously from zero to uniform velocity. In other words, it is found that the surface of the body is covered by a coat comprising a thin layer where the velocity gradient is large. This layer forms a zone of reduced velocity, causing vortices, called a wake, to be cast off downstream of the body.

We notice the existence of boundary layers daily in various ways. For example, everybody experiences the feeling of the wind blowing (as shown in Fig. 6.18) when standing in a strong wind at the seaside; however, by stretching out on the beach much less wind is felt. In this case the boundary layer on the ground extends to as much as 1 m or more, so the nearer the

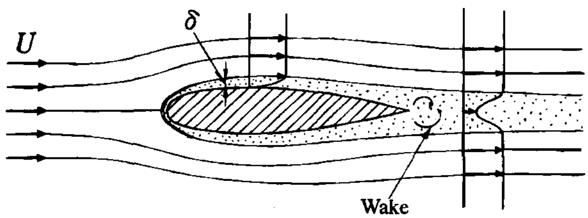


Fig. 6.17 Boundary layer around body

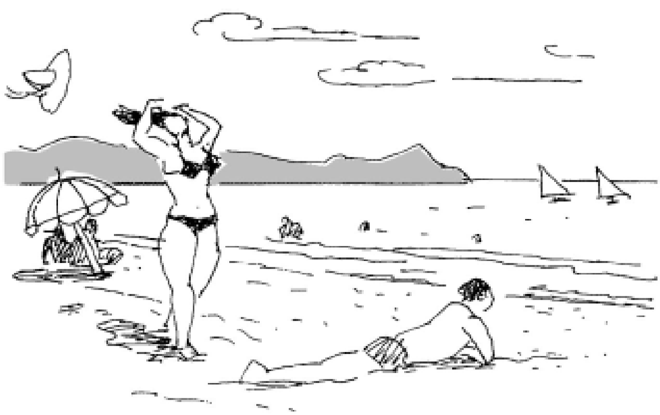


Fig. 6.18 Man lying down is less affected by the coastal breeze than woman standing up

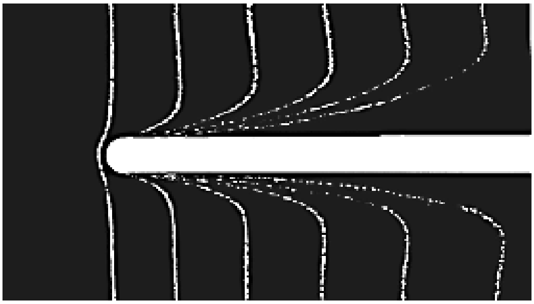


Fig. 6.19 Development of boundary layer on a flat plate (thickness 5 mm) in water, velocity 0.6 m/s

ground the smaller the wind velocity. The velocity u within the boundary layer increases with the distance from the body surface and gradually approaches the velocity of the main flow. Since it is difficult to distinguish the boundary layer thickness, the distance from the body surface when the velocity reaches 99% of the velocity of the main flow is defined as the boundary layer thickness δ. The boundary layer continuously thickens with the distance over which it flows. This process is visualized as shown in Fig. 6.19. This thickness is less than a few millimetres on the frontal part of a high-speed aeroplane, but reaches as much as 50 cm on the rear part of an airship.

When the flow distribution and the drag are considered, it is useful to use the following displacement thickness δ^* and momentum thickness θ instead of δ.

$$U\delta^* = \int_0^\infty (U - u)\mathrm{d}y \tag{6.53}$$

$$\rho U^2\theta = \rho \int_0^\infty u(U - u)\mathrm{d}y \tag{6.54}$$

δ^* is the position which equalises two zones of shaded portions in Fig. 6.20(a). It corresponds to an amount δ^* by which, owing to the development

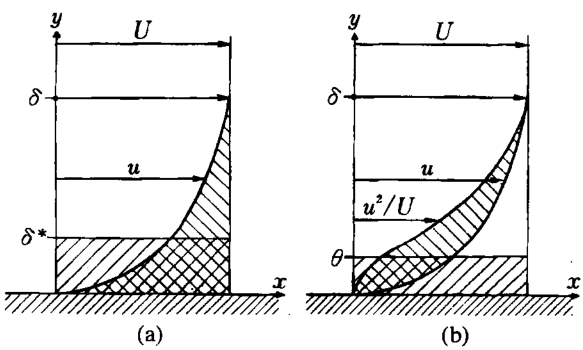


Fig. 6.20 Displacement thickness (a) and momentum thickness (b)

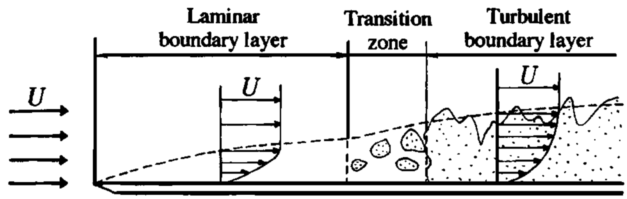


Fig. 6.21 Boundary layer on a flat board surface

of the boundary layer, a body appears larger to the external flow compared with the case where the body is an inviscid fluid. Consequently, in the case where the state of the main flow is approximately obtained as inviscid flow, a computation which assumes the body to be larger by δ^* produces a result nearest to reality. Also, the momentum thickness θ equates the momentum decrease per unit time due to the existence of the body wall to the momentum per unit time which passes at velocity U through a height of thickness θ. The momentum decrease is equivalent to the force acting on the body according to the law of momentum conservation. Therefore the drag on a body generated by the viscosity can be obtained by using the momentum thickness.

Consider the case where a flat plate is placed in a uniform flow. The flow velocity is zero on the plate surface. Since the shearing stress due to viscosity acts between this layer and the layer immediately outside it, the velocity of the outside layer is reduced. Such a reduction extends to a further outside layer and thus the boundary layer increases its thickness in succession, beginning from the front end of the plate as shown in Fig. 6.21.

In this manner, an orderly aligned sheet of vorticity diffuses. Such a layer is called a laminar boundary layer, which, however, changes to a turbulent boundary layer when it reaches some location downstream.

This transition to turbulence is caused by a process in which a very minor disturbance in the flow becomes more and more turbulent until at last it makes the whole flow turbulent. The transition of the boundary layer therefore does not occur instantaneously but necessitates some length in the direction of the flow. This length is called the transition zone. In the transition zone the laminar state and the turbulent state are mixed, but the further the flow travels the more the turbulent state occupies until at last it becomes a turbulent boundary layer.

The velocity distributions in the laminar and turbulent boundary layers are similar to those for the flow in a pipe.

6.5.2 Equation of motion of boundary layer

Consider an incompressible fluid in a laminar boundary layer. Each component of the equation of motion in the y direction is small compared with that in the x direction, while $\partial^2 u/\partial x^2$ is also small compared with $\partial^2 u/\partial y^2$. Therefore, the Navier–Stokes equations (6.12) simplify the following equations:

$$\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{\partial p}{\partial x}+\mu\frac{\partial^2 u}{\partial y^2} \tag{6.55}$$

$$\frac{\partial p}{\partial y}=0 \tag{6.56}$$

The continuity equation is as follows:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{6.57}$$

Equations (6.55)–(6.57) are called the boundary layer equations of laminar flow.

For a steady-state turbulent boundary layer, with similar considerations, the following equations result:

$$\rho\left(\bar{u}\frac{\partial \bar{u}}{\partial x}+\frac{\partial \bar{u}}{\partial y}\right)=-\frac{\partial \bar{p}}{\partial x}+\frac{\partial \tau}{\partial y} \tag{6.58}$$

$$\tau=\mu\frac{\partial \bar{u}}{\partial y}-\rho\overline{u'v'} \tag{6.59}$$

$$\frac{\partial \bar{p}}{\partial y}=0 \tag{6.60}$$

$$\frac{\partial \bar{u}}{\partial x}+\frac{\partial \bar{v}}{\partial y}=0 \tag{6.61}$$

Equations (6.58)–(6.61) are called the boundary layer equations of turbulent flow.

6.5.3 Separation of boundary layer

In a flow where the pressure decreases in the direction of the flow, the fluid is accelerated and the boundary layer thins. In a contraction flow, the pressure has such a negative (favourable) gradient that the flow stabilises while the turbulence gradually decreases.

In contrast, things are quite different in a flow with a positive (adverse) pressure gradient where the pressure increases in the flow direction, such as a divergent flow or flow on a curved wall as shown in Fig. 6.22. Fluid far off the wall has a large flow velocity and therefore large inertia too. Therefore, the flow can proceed to a downstream location overcoming the high pressure downstream. Fluid near the wall with a small flow velocity, however, cannot overcome the pressure to reach the downstream location because of its small inertia. Thus the flow velocity becomes smaller and smaller until at last the velocity gradient becomes zero. This point is called the separation point of the flow. Beyond it the velocity gradient becomes negative to generate a flow reversal. In this separation zone, more vortices develop than in the ordinary boundary layer, and the flow becomes more turbulent. For this reason the energy loss increases. Therefore, an expansion flow is readily destabilised with a large loss of energy.

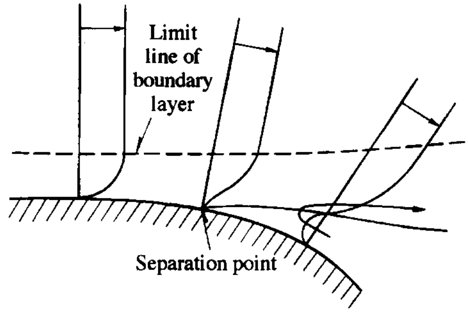


Fig. 6.22 Separation of boundary layer

6.6 Theory of lubrication

As shown in Fig. 6.23, consider two planes with a wedge-like gap containing an oil film between them. Assume that the upper plane is stationary and of length l inclined to the x axis by α, and that the lower plane is an infinitely long plane moving at constant velocity U in the x direction. By the movement of the lower plane the oil stuck to it is pulled into the wedge. As a result, the internal pressure increases to push up the upper plane so that the two planes do not come into contact. This is the principle of a bearing. In this flow, since the oil-film thickness is small in comparison with the length of plane in the flow direction, the flow is laminar where the action of viscosity is very dominant. Therefore, by considering it in the same way as a flow between parallel planes (see Section 6.3.1), the following equation is obtained from eqn (6.12):

$$\frac{\mathrm{d}p}{\mathrm{d}x} = \mu \frac{\partial^2 u}{\partial y^2} \tag{6.62}$$

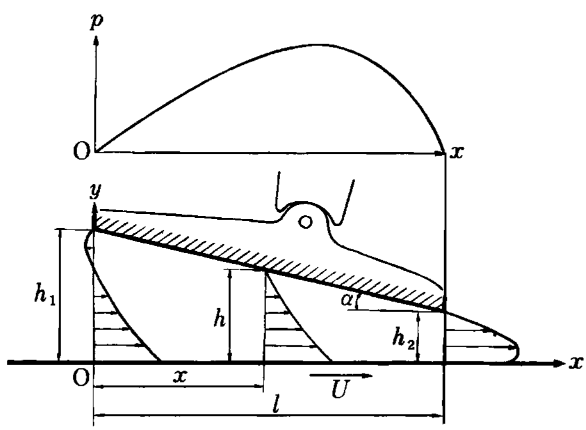


Fig. 6.23 Flow and pressure distribution between inclined planes (slide bearing)

In this case, the pressure p is a function of x only, so the left side is an ordinary differential.

Integrate eqn (6.62) and use boundary conditions $u = U, y = h$ and $u = 0$ at $y = 0$. Then

$$u = U\left(1 - \frac{y}{h}\right) - \frac{\mathrm{d}p}{\mathrm{d}x}\frac{h^2}{2\mu}\frac{y}{h}\left(1 - \frac{y}{h}\right) \tag{6.63}$$

The flow rate Q per unit width passing here is

$$Q = \int_0^h u\,\mathrm{d}y \tag{6.64}$$

Substituting eqn (6.63) into (6.64),

$$Q = \frac{Uh}{2} - \frac{h^3}{12\mu}\frac{\mathrm{d}p}{\mathrm{d}x} \tag{6.65}$$

From the relation $(h_1 - h_2)/l = \alpha$,

$$h = h_1 - \alpha x \tag{6.66}$$

Substituting the above into eqn (6.65),

$$\frac{\mathrm{d}p}{\mathrm{d}x} = \frac{6\mu U}{(h_1 - \alpha x)^2} - \frac{12\mu Q}{(h_1 - \alpha x)^3} \tag{6.67}$$

Integrating eqn (6.67),

$$p = \frac{6\mu U}{\alpha(h_1 - \alpha x)} - \frac{6\mu Q}{\alpha(h_1 - \alpha x)^2} + c \tag{6.68}$$

Assume $p = 0$ when $x = 0, x = l$, so

$$Q = \frac{h_1 h_2}{h_1 + h_2}U \quad c = -\frac{6\mu U}{\alpha(h_1 + h_2)}$$

Equation (6.68) becomes as follows:

$$p = \frac{6\mu U(h - h_2)}{(h_1 + h_2)h^2}x \tag{6.69}$$

From eqn (6.69), since $h > h_2$, $p > 0$. Consequently, it is possible to have the upper plane supported above the lower plane. This pressure distribution is illustrated in Fig. 6.23. By integrating this pressure, the supporting load P per unit width of bearing is obtained:

$$P = \int_0^l p\,\mathrm{d}x = \frac{6\mu U l^2}{(h_1 - h_2)^2}\left[\log\left(\frac{h_1}{h_2}\right) - 2\frac{h_1 - h_2}{h_1 + h_2}\right] \tag{6.70}$$

From eqn (6.70), the force P due to the pressure reaches a maximum when $h_1/h_2 = 2.2$. At this condition P is as follows:

$$P_{\max} = 0.16\frac{\mu U l^2}{h_2^2} \tag{6.71}$$

This slide bearing is mostly used as a thrust bearing. The theory of lubrication above was first analysed by Reynolds.

The principle of the journal bearing is almost the same as the above case. However, since oil-film thickness h is not expressed by the linear equation of x as shown by eqn (6.66), the computation is a little more complicated. This analysis was performed by Sommerfeld and others.

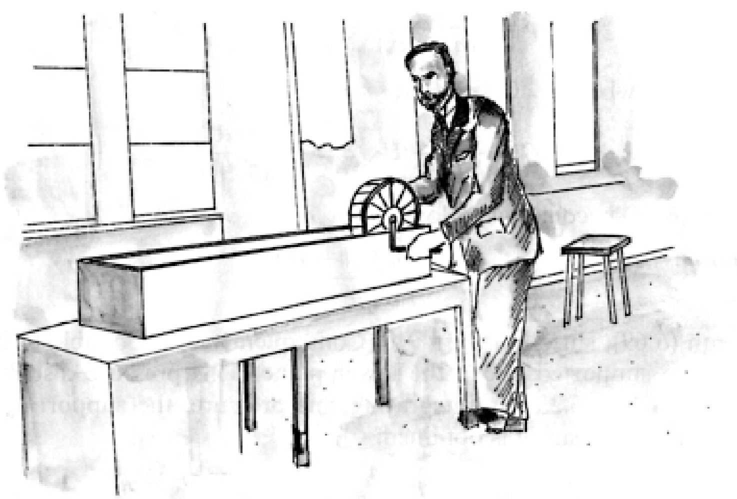
Homer sometimes nods

This is an example in which even such a great figure as Prandtl made a wrong assumption. On one occasion, under the guidance of Prandtl, Hiementz set up a tub to make an experiment for observing a separation point on a cylinder surface. The purpose was to confirm experimentally the separation point computed by the boundary layer theory. Against his expectation, the flow observed in the tub showed violent vibrations.

Hearing of the above vibration, Prandtl responded, 'It was most likely caused by the imperfect circularity of the cylinder section shape.'

Nevertheless, however carefully the cylinder was reshaped, the vibrations never ceased.

Kármán, then an assistant to Prandtl, assumed there was some essential natural phenomenon behind it. He tried to compute the stability of vortex alignment. Summarising the computation over the weekend, he showed the summary to Prandtl on Monday for his criticism. Then, Prandtl told Kármán, 'You did a good job. Make it up into a paper as quickly as possible. I will submit it for you to the Academy.'



A bird stalls

Kármán hit upon the idea of making a bird stall by utilising his knowledge in aerodynamics. When he was standing on the bank of Lake Constance with a piece of bread in his hand, a gull approached him to snatch the bread. Then he slowly withdrew his hand, and the gull tried to slow down its speed for snatching. To do this, it had to increase the lift of its wings by increasing their attack angles. In the course of this, the attack angles probably exceeded their effective limits. Thus the gull sometimes lost its speed and fell (see 'stall', page 164).

Benarl and Kármán

Kármán's train of vortices has been known for so long that it is said to appear on a painting inside an ancient church in Italy. Even before Kármán, however, Professor Henry Benarl (1874–1939) of a French university observed and photographed this train of vortices. Therefore, Benarl insisted on his priority in observing this phenomenon at a meeting on International Applied Dynamics. Kármán responded at the occasion 'I am agreeable to calling Henry Benarl Street in Paris what is called Kármán Street in Berlin and London.' With this joke the two became good friends.

6.7 Problems

1. Show that the continuity equation in the flow of a two-dimensional compressible fluid is as follows:

$$\frac{\partial \rho}{\partial t}+\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}=0$$

2. If the flow of an incompressible fluid is axially symmetric, develop the continuity equation using cylindrical coordinates.
3. If flow is laminar between parallel plates, derive equations expressing (a) the velocity distribution, (b) the mean and maximum velocity, (c) the flow quantity, and (d) pressure loss.

4. If flow is laminar in a circular tube, derive equations expressing (a) the velocity distribution, (b) the mean and maximum velocity, (c) the flow quantity, and (d) pressure loss.

5. If flow is turbulent in a circular tube, assuming a velocity distribution $u = u_{max}(y/r_0)^{1/7}$, obtain (a) the relationship between the mean velocity and the maximum velocity, and (b) the radius of the fluid flowing at mean velocity.

6. Water is flowing at a mean velocity of 4 cm/s in a circular tube of diameter 50 cm. Assume the velocity distribution $u = u_{max}(y/r_0)^{1/7}$. If the shearing stress at a location 5 cm from the wall is 5.3×10^{-3} N/m^2, compute the turbulent kinematic viscosity and the mixing length. Assume that the water temperature is 20°C and the mean velocity is 0.8 times the maximum velocity.

7. Consider a viscous fluid flowing in a laminar state through the annular gap between concentric tubes. Derive an equation which expresses the amount of flow in this case. Assume that the inner diameter is d, the gap is h, and $h \ll d$.

8. Oil of 0.09 Pa s (0.9 P) fills a slide bearing with a flat upper face of length 60 cm. A load of 5×10^2 N per 1 cm of width is desired to be supported on the upper surface. What is the maximum oil-film thickness when the lower surface moves at a velocity of 5 m/s?

9. Show that the friction velocity $\sqrt{\tau_0/\rho}$ (τ_0: shearing stress of the wall; ρ: fluid density) has the dimension of velocity.

10. The piston shown in Fig. 6.24 is moving from left to right in a cylinder at a velocity of 6 m/s. Assuming that lubricating oil fills the gap between the piston and the cylinder to produce an oil film, what is the friction force acting on the moving piston? Assume that the kinematic viscosity of oil $\nu = 50$ cSt, specific gravity = 0.9, diameter of cylinder $d_1 = 122$ mm, diameter of piston $d_2 = 125$ mm, piston length $l = 160$ mm, and that the pressure on the left side of the piston is higher than that on the right side by 10 kPa.

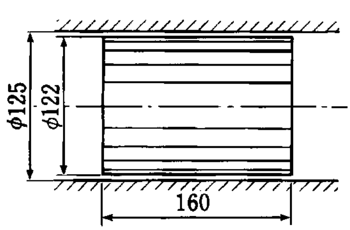


Fig. 6.24